J. Appl. Maths Mechs Vol. 57, No. 3, pp. 525–535, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

SHEAR INSTABILITY OF A TWO-PHASE PERIODIC STRUCTURE†

M. P. LAZAREV

Moscow

(*Received* 13 *April* 1992)

A periodic structure consisting of alternating layers of two different elastic solid phases of the same chemical substance is considered. It is assumed that the phase transformation is accompanied by a small affine "natural" deformation (ND) which turns the reference configuration of the "plus" phase into the reference configuration of the "minus" phase. The gradients of the displacements produced by the ND are specified by the constant tensor Δ_{ij}. The interphase boundaries between different phases are assumed to be coherent [1], i.e. such that the transformation preserves particle adjacency. The stability of an equilibrium system in which the phases are in homogeneous states with constant displacement gradients κ_{ij}^{+} relative to their reference configurations is studied. The connection between the tensors κ_{ij}^{+} and κ_{ij}^{-} is given by relations (1) and (2) of [2]. From these formulae it follows that for a periodic structure the states in layers of the same phase are identical. The states in phase "minus" layers are uniquely defined by the given state of phase "plus", the ND transformation and the known elastic moduli of the phases.

Unlike the special case considered previously [3], here we consider the general situation when both tensors κ_{ij}^{+}, Δ_{ij} and the layer thicknesses are arbitrarily specified. The influence of these parameters on the stability of the two-phase periodic structure is analysed. A "shear instability" effect is observed, which is a distinctive feature of heterogeneous layered structures with coherent interphase boundaries.

1. STATEMENT OF THE PROBLEM

WE WILL use a Cartesian system of coordinates (x, z). Let $z\in(0,\ 2H_+)$ be the "plus" phase layer, and $z\in(-2H_-,\ 0)$ be the "minus" phase layer. The problem of the stability of piecewise-homogeneous configurations with plane interphase boundaries reduces [1] to an analysis of the sign-definiteness of the spectral eigenvalues π of a system of linear homogeneous differential equations with constant coefficients. Special boundary conditions for this system contain information on the stressed states of the phases. If for the equilibrium configuration under consideration at least one negative eigenvalue π of the spectral problem exists, that configuration is unstable. In the case of isotropic incompressible phases in a state of plane deformation [4] (the plane of deformation being perpendicular to the layers), the spectral problem has the following form [3]

$$c_{\perp\pm}^2\left(\frac{\partial^2 a_1^\pm}{\partial x^2}+\frac{\partial^2 a_1^\pm}{\partial z^2}\right)-m^{-1}\frac{\partial p'_\pm}{\partial x}+\pi a_1^\pm=0 \tag{1.1}$$

$$c_{\perp\pm}^2\left(\frac{\partial^2 a_2^\pm}{\partial x^2}+\frac{\partial^2 a_2^\pm}{\partial z^2}\right)-m^{-1}\frac{\partial p'_\pm}{\partial z}+\pi a_2^\pm=0,\quad \frac{\partial a_1^\pm}{\partial x}+\frac{\partial a_2^\pm}{\partial z}=0$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 3, pp. 125–136, 1993.

at the interphase boundary

$$\left[\frac{\partial a_1}{\partial x}-\beta\frac{\partial a_2}{\partial x}\right]=0,\quad \left[-m^{-1}p'+2c_\perp^2\frac{\partial a_2}{\partial z}\right]=0$$

$$\left[c_\perp^2\left(\frac{\partial a_1}{\partial z}+\frac{\partial a_2}{\partial z}\right)+\alpha\frac{\partial a_2}{\partial x}\right]=0 \tag{1.2}$$

$$\alpha\frac{\partial a_1^-}{\partial x}-\beta c_{\perp+}^2\left(\frac{\partial a_1^+}{\partial z}+\frac{\partial a_2^+}{\partial x}\right)-2c_{\perp+}^2\frac{\partial a_2^+}{\partial z}+m^{-1}p'_+=0$$

$$\alpha\equiv\frac{[2c_\perp^2(\kappa_{11}-\kappa_{22})]}{[\kappa_{22}+\Delta_{22}]}=-m^{-1}\Delta^{-1}((\sigma_{11}^+-\sigma_{22}^+)(1-\chi^{-1})+4\mu_-\Delta_{11})$$

$$\beta\equiv\frac{[\kappa_{12}+\Delta_{12}]}{[\kappa_{22}+\Delta_{22}]}=\Delta^{-1}(2\Delta_{(12)}+(\chi-1)\sigma_{12}^+/\mu_+) \tag{1.3}$$

$$\Delta\equiv\Delta_{11}+\Delta_{22},\quad \chi\equiv\mu_+/\mu_-,\quad \sigma_{ij}\equiv-p\delta_{ij}+2\mu\kappa_{(ij)},\quad i,j=1,2.$$

Here $c_\perp$ is the velocity of transverse three-dimensional waves, m is the mass density in the initial single-phase configuration, $a_i(x, z)$ are the perturbations (variations) of the particle displacement field, $i=1$ corresponds to orientation along the x axis, $i=2$ corresponds to orientation along the z axis, $p'(x, z)$ is the limiting value of the quantity $-\lambda(\partial a_1/\partial x+\partial a_2/\partial z)$ for incompressible phases with λ being the Lamé coefficient, the $[\cdot]$ denotes a jump: $[a]\equiv a_+-a_-$, μ is the shear modulus, and α and β are parameters of the stressed state of the two-phase equilibrium configuration.

Consider the stability of a periodic structure with respect to periodic perturbations of the displacement fields $a_i^\pm$ and pressure $p'_\pm$, i.e.

$$a_i(x,z)=a_i(x,z\pm2(H_++H_-))$$
$$p'(x,z)=p'(x,z\pm2(H_++H_-)) \tag{1.4}$$

A "periodic cell" of thickness $2(H_++H_-)$ and composed of two neighbouring layers arises naturally in the analysis of the equilibrium and stability of this system. All properties of the system are repeated when going from one cell to the next. Solutions of the spectral problem on the external boundaries of the cell are subject to matching conditions.

We say that the two-phase periodic structure is stable if the solutions $a_i(x, z)$ and $p'(x, z)$ of spectral problem (1.1), (1.2) that are periodic along the z axis and are oscillatory in nature along the interphase boundaries correspond to non-negative eigenvalues π. If there are negative eigenvalues, the corresponding system is said to be unstable.

2. SYMMETRIC AND ANTISYMMETRIC PERTURBATIONS

The general solution of the system of differential equations (1.1) in the cell $z\in(-2H_-, 2H_+)$ can be represented in the form of the sum of symmetric perturbations $f_i^\pm$, $p'_{f\pm}$ and anti-symmetric perturbations $g_i^\pm$, $p'_{g\pm}$

$$a_i^\pm=f_i^\pm+g_i^\pm,\quad p'_\pm=p'_{f\pm}+p'_{g\pm} \tag{2.1}$$

$$f_1^\pm=i(F_1^\pm\operatorname{ch}k(z\mp H_\pm)+F_2^\pm\xi_\pm\operatorname{ch}k\xi_\pm(z_\mp H_\pm))\exp(-ikx)$$

$$f_2^\pm=-(F_1^\pm\operatorname{sh}k(z\mp H_\pm)+F_2^\pm\operatorname{sh}k\xi_\pm(z\mp H_\pm))\exp(-ikx)$$

$$p'_{p\pm} = -qmkF_1^{\pm} \operatorname{ch} k(z \mp H_\pm)\exp(-ikx), \quad \xi_\pm \equiv (1 - qc_{\perp\pm}^{-2})^{1/2}, \quad q \equiv \pi k^{-2}$$

Here k is the real wave number and $F_{1,2}^{\pm}$ are constants. For antisymmetric perturbations $g_i^{\pm}$, $p'_{g\pm}$ the cosines and sines in the formulae for $f_i^{\pm}$, $p'_{f\pm}$ are exchanged and instead of the constants $F_{1,2}^{\pm}$ we use $G_{1,2}^{\pm}$.

3. DISPERSION RELATIONS AND THE EQUATION OF NEUTRAL STABILITY

Substituting (2.1) into the interphase boundary conditions (1.2) we obtain a linear homogeneous system of algebraic equations in $F_{1,2}^{\pm}$, $G_{1,2}^{\pm}$. The condition for a non-trivial solution (2.1) of the spectral problem (1.1), (1.2) to exist is that the determinant of the matrix D for this system should vanish.

Computing the determinant of D we obtain a dispersion relation for finding the spectral eigenvalue π

$$\|D\| = A\beta^4 + B\beta^2 + C = 0 \tag{3.1}$$

$$A \equiv R_t \cdot R_c, \quad C \equiv d_f \cdot d_g, \quad B \equiv 2B_{1t}B_{1c} + B_{2t}B_{3c} + B_{3t}B_{2c}$$

$$R_t \equiv q(S_-^2 \operatorname{cth} h_- + S_+^2 \operatorname{cth} h_+ - 4c_{\perp+}^4\xi_+ \operatorname{cth} h_+\xi_+ - 4c_{\perp-}^4\xi_- \operatorname{cth} h_-\xi_-)$$

$$B_{1t} \equiv (S_+R_+ \operatorname{th} h_+\xi_+ - 2c_{\perp+}^2\xi_+Q_+ \operatorname{th} h_+)(R_- \operatorname{th} h_-\xi_- - Q_-\xi_- \operatorname{th} h_-) -$$
$$-(2c_{\perp+}^2\xi_+ \operatorname{th} h_+ - S_+ \operatorname{th} h_+\xi_+)(Q_-^2\xi_- \operatorname{th}_- h_- - R_-^2 \operatorname{th} h_-\xi_-) -$$
$$-q \operatorname{th} h_+ \operatorname{th} h_+\xi_+(2c_{\perp-}^2\xi_-R_- - S_-Q_-), \quad B_{2t} \equiv q^2\alpha^2\xi_- \operatorname{th} h_+ \operatorname{th} h_+\xi_+ -$$
$$-(S_+^2 \operatorname{th} h_+\xi_+ - 4c_{\perp+}^4\xi_+ \operatorname{th} h_+)(R_-^2 \operatorname{th} h_-\xi_- - Q_-^2\xi_- \operatorname{th} h_-)$$

$$B_{3t} \equiv (2c_{\perp-}^2Q_-\xi_- \operatorname{th} h_- - S_-R_- \operatorname{th} h_-\xi_-)(\xi_+ \operatorname{th} h_+ - \operatorname{th} h_+\xi_+) +$$
$$+(S_+R_+ \operatorname{th} h_+\xi_+ - 2c_{\perp+}^2\xi_+Q_+ \operatorname{th} h_+)(\operatorname{th} h_-\xi_- - \xi_- \operatorname{th} h_-) +$$
$$+(Q_+\xi_+ \operatorname{th} h_+ - R_+ \operatorname{th} h_+\xi_+)(S_- \operatorname{th} h_-\xi_- - 2c_{\perp-}^2\xi_- \operatorname{th} h_-) +$$
$$+(R_- \operatorname{th} h_-\xi_- - Q_-\xi_- \operatorname{th} h_-)(2c_{\perp+}^2\xi_+ \operatorname{th} h_+ - S_+ \operatorname{th} h_+\xi_+) -$$
$$-q^2\xi_+ \operatorname{th} h_- \operatorname{th} h_-\xi_- - q^2\xi_- \operatorname{th} h_+ \operatorname{th} h_+\xi_+$$

$$d_f \equiv q(\xi_+\xi_-(Q_-^2 \operatorname{th} h_- + Q_+^2 \operatorname{th} h_+) - \xi_+R_-^2 \operatorname{th} h_-\xi_- - \xi_-R_+^2 \operatorname{th} h_+\xi_+)$$

$$Q_\pm \equiv 2c_{\perp\pm}^2 + \alpha, \quad R_\pm \equiv 2c_{\perp\pm}^2 - q + \alpha, \quad S_\pm \equiv 2c_{\perp\pm}^2 - q, \quad h_\pm \equiv kH_\pm$$

The formulae for R_c, B_{ic} $(i = 1, 2, 3)$ and d_g are obtained from formulae for R_t, B_{it} and d_f by interchanging the tangents and cotangents.

Expanding the indeterminacy $(q^{-4}\|D\|)_{q=0}$, we arrive at the equation of neutral stability

$$(2c_{\perp+}c_{\perp-})^2(q^{-4}\|D\|)_{q=0} = a\beta^4 + b\beta^2 + d = 0 \tag{3.2}$$

$$a \equiv a_ta_c, \quad b \equiv 2b_{1t}b_{1c} + b_{2t}b_{3c} + b_{3t}b_{2c}, \quad d \equiv d_td_c$$

$$a_t \equiv l_- / \sqrt{\chi} + l_+\sqrt{\chi}, \quad \bar{\alpha} \equiv \alpha / (2c_{\perp+}c_{\perp-})$$

$$b_{1t} \equiv -\bar{\alpha}(\chi n_+l_+ + \operatorname{th}^2 h_+ + t_+t_-) + n_+t_-\sqrt{\chi} - n_-t_+ / \sqrt{\chi}$$

$$b_{2t} \equiv \bar{\alpha}^2(\chi n_+l_- + \operatorname{th}^2 h_+) - 2\bar{\alpha}t_-n_+\sqrt{\chi} - n_+n_-$$

$$b_{3t} \equiv -(\chi^{-1}l_+n_- + \chi l_-n_+ + \mathrm{th}^2 h_- + \mathrm{th}^2 h_+ + 2t_+t_-)$$
$$d_t \equiv -\bar{\alpha}^2(l_-\sqrt{\chi} + l_+ / \sqrt{\chi}) + 2\bar{\alpha}(t_+ + t_-) + n_- / \sqrt{\chi} + n_+\sqrt{\chi}$$
$$t_\pm \equiv h_\pm(1 - \mathrm{th}^2 h_\pm), \quad l_\pm \equiv \mathrm{th}\, h_\pm - t_\pm, \quad n_\pm \equiv \mathrm{th}\, h_\pm + t_\pm$$

The formulae for a_c, b_{ic} $(i = 1, 2, 3)$ and d_c are obtained from formulae (3.2) by replacing the tangents by cotangents.

For the two-phase periodic structure to be stable Eq. (3.1) must not have negative roots q.

We shall study the influence of the parameters α, β and $h_\pm$ on the stability of two-phase equilibrium.

4. THE $\beta = 0$ CASE

In this case symmetric and antisymmetric perturbations are in themselves separate solutions of the spectral problem (1.1), (1.2). (This assertion does not hold when $\beta \neq 0$.)

If the condition $\beta = 0$ holds in the "plus" phase layers, and $\bar{\alpha} \in (-\infty, \bar{\alpha}_1^f) \cup (\bar{\alpha}_2^f, +\infty)$ $(\bar{\alpha} \in (-\infty, \bar{\alpha}_1^g) \cup (\bar{\alpha}_2^g, +\infty))$, then the equilibrium of the structure is unstable with respect to symmetric (antisymmetric) perturbations. Here $\bar{\alpha}_{1,2}^f$ are the roots of the equation $d_t = 0$, and $\bar{\alpha}_{1,2}^g$ are the roots of the equation $d_c = 0$.

The parameter α defined by (1.3) describes the "non-hydrostatic jump" of the stresses in the phases across the coherent boundary. The growth in absolute magnitude of this jump, as was shown above, leads to instability in the periodic structure. This result agrees with those obtained in [2, 5].

We shall indicate the unstable domains for some physically interesting limiting cases.

We denote by s the concentration of the "minus" phase in the periodic cell: $s = h_-(h_+ + h)^{-1}$, $0 \leqslant s \leqslant 1$. Then $h_- = sh$, $h_+ = (1-s)h$, where $h = h_+ - h_-$.

If $0 < s < 1$ and $h \to +\infty$, then $\bar{\alpha}_1^{f,g} \to -1$ and $\bar{\alpha}_2^{f,g} \to 1$.

Suppose $h \to 0$ and $0 < s < 1$ (that physically corresponds to the case of large perturbation lengths k^{-1} and finite layer thicknesses $H_\pm$ or the case of finite perturbation lengths and small layer thicknesses). The structure will be unstable with respect to symmetric (antisymmetric) perturbations if $\bar{\alpha} \in (-\infty, \bar{\alpha}_{01}^f)$ $(\bar{\alpha} \in (-\infty, \bar{\alpha}_{01}^g) \cup (0, \infty))$ since

$$\begin{gathered}\bar{\alpha}_{01}^f \equiv \lim \bar{\alpha}_1^f = -(s/\sqrt{\chi} + (1-s)\sqrt{\chi}), \quad \lim \bar{\alpha}_2^f = +\infty \\ \bar{\alpha}_{01}^g \equiv \lim \bar{\alpha}_1^g = -(s/\sqrt{\chi} + (1-s)\sqrt{\chi})^{-1}, \quad \lim \bar{\alpha}_2^g = 0, (h \to 0).\end{gathered} \tag{4.1}$$

When $s \to 0$, $h \sim 1$ (i.e. $h_+ \sim 1$, $h_- \sim 0$), that corresponds to a two-phase structure with thin periodic nucleation layers, the system will be unstable with respect to symmetric (antisymmetric) perturbations if $\bar{\alpha} \in (-\infty, -\sqrt{\chi}) \cup (\sqrt{\chi}(\sinh 2h_+ + 2h_+)(\sinh 2h_+ - 2h_+)^{-1}, +\infty)$, $(\bar{\alpha} \in (-\infty, -1\sqrt{\chi}) \cup (0, +\infty))$.

If $s \to 0$, $h \to +\infty$ (i.e. $h_+ \to +\infty$, $h_- \ll +\infty$), we have the solution of the stability problem for a plane nucleation layer of a new "minus" phase in an infinite elastic matrix of the original "plus" phase [5].

The roots $\bar{\alpha}_{1,2}^f$ and $\bar{\alpha}_{1,2}^g$ were calculated on a computer for various values of the parameters s, h and χ. Figure 1 shows the behaviour of $\bar{\alpha}_{1,2}^f(s)$, $\bar{\alpha}_{1,2}^g(s)$ for $h = 2$, $\chi = 0.1$, and also of $\bar{\alpha}_{01}^f(s)$ and $\bar{\alpha}_{01}^g(s)$ for $\chi = 0.1$. ($\bar{\alpha}_2^f \to +\infty$ and $\bar{\alpha}_2^g \to 0$ as $h \to 0$.) The dashed lines show $\bar{\alpha}^f$ and the solid lines show $\bar{\alpha}^g$. Graphs of the functions $\bar{\alpha}_{1,2}^f(h)$ and $\bar{\alpha}_{1,2}^g(h)$ were given in [3].

5. EXAMPLE

Consider the case of a cubic expansion/compression ND $(\Delta_{ij} = \delta\delta_{ij})$ with $\chi = 0.1$, $\sigma_{12} = 0$. If $\delta > 0$ $(\delta < 0)$ then the phase transformation is accompanied by cubic expansion (compression) of particles of the less stiff "plus" phase and the formation of the stiffer "minus" phase.

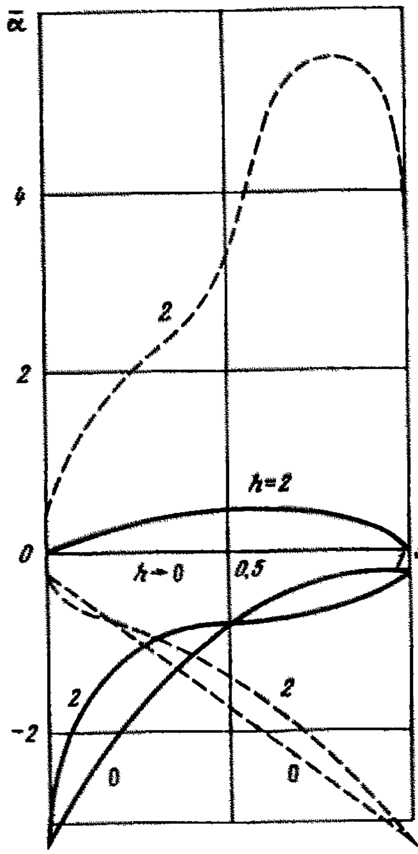

FIG. 1.

5.1. Suppose that in the less stiff phase the principal stresses in the direction of the tangent to the interphase boundary are zero ($\sigma_{11}^{+}=0$). Here the same stresses in the more rigid phase are given by formula (2) of [2]: $\sigma_{11}^{-}=-9\sigma_{22}^{+}-4\mu_{-}\delta$. The critical values of the stress σ_{22} ($\sigma_{22}^{+}=\sigma_{22}^{-}=\sigma_{22}$) and the critical values of the parameter $\bar\alpha$ are related by formula (1.3)

$$\bar\alpha^{*}=-\tfrac{9}{4}\delta^{-1}(\mu_{+}\mu_{-})^{-1/2}\sigma_{22}^{*}-\sqrt{10}$$

(Asterisks identify critical values.)

If the less stiff phase is in a hydrostatic state ($\sigma_{11}^{+}=\sigma_{22}=0$) then $\bar\alpha=-\sqrt{10}$. In this case we have instability with respect to both symmetric and antisymmetric perturbations ([3], Fig. 1) when $s=\tfrac{1}{2}$. A more detailed numerical analysis shows that this is also true for all values of the concentration s. For a cubic expansion (compression) ND the conditions necessary for stability can only be satisfied as a result of the application of compressive (extensive) principal stresses along the z axis: $\sigma_{22}>0$ ($\sigma_{22}<0$).

5.2. Suppose that in the stiffer phase the principal stresses in the tangential direction to the interphase boundary vanish ($\sigma_{11}^{-}=0$). Here $\sigma_{11}^{+}=0.9\sigma_{22}+4\mu_{+}\delta$ ([2], formula (2)). The critical values of $\bar\alpha$ and σ_{22} are related by the equation

$$\bar\alpha^{*}=-0.2025\delta^{-1}(\mu_{+}\mu_{-})^{-1/2}\sigma_{22}^{*}-\sqrt{0.1} \tag{5.1}$$

If the stiffer phase is in a hydrostatic state, then $\bar\alpha=-\sqrt{0.1}$. In this case the necessary conditions for stability are satisfied by both types of perturbation (Fig. 1). Numerical analysis shows that this is also true for any s and h.

Consider the upper branches of the $\bar\alpha^{*}$ curve (i.e. $\bar\alpha_2^{(*)}$) in Fig. 1. If $\delta>0$ ($\delta<0$), these branches correspond to the critical compressive (extensive) stress function $\sigma_{22}^{*}(s)$. In Fig. 1 these functions have clear-cut maxima for both types of perturbation. Here the maxima are displaced towards the $s=1$ axis. This shows the onset of the stabilizing influence of increasing concentration of the stiffer phase. When $s\to 0$ or $s\to 1$ the margin of stability diminishes, i.e. a considerable reduction in the thickness of one of the phase layers destabilizes the system.

The lower branches of $\bar{\alpha}^{*}$ (i.e. $\bar{\alpha}_1^{f,g}$) are represented by monotonic functions of s when $h=2$. As the concentration of the stiffer ("minus") phase increases the absolute value of the critical stresses $\sigma_{22}^{*}(s)$ for symmetric perturbations increases monotonically, and decreases for antisymmetric perturbations (see formula (5.1)).

6. ANALYSIS OF THE SOLUTION WHEN $\beta\neq 0$. THE "SHEAR INSTABILITY" EFFECT

First we will explain the physical meaning of β. We put $\varepsilon^{+}=\kappa_{(12)}^{+}$ and $\varepsilon^{-}=\Delta_{(12)}+\kappa_{(12)}^{-}$. Consider straight material lines passing through the initial single-phase configuration (which is identified with the reference configuration of the "plus" phase) perpendicular to the pre-image surface of the interphase boundary. The quantity $\Delta_{(12)}$ defines the angle of deviation of the material lines in the reference configuration of the "minus" phase from the position in the initial configuration. The quantities $\kappa_{(12)}^{\pm}$ define the angle of deviation of the material lines in the actual configuration from their positions in the reference configurations. The quantities $\varepsilon^{\pm}$ define (respectively for each phase) the angles of deviation of the above-mentioned material lines in the actual configuration from their initial geometrical positions. Thus the quantity $[\varepsilon]$ has the geometrical meaning of the jump in the angles of inclination of the above-mentioned material lines in the actual two-phase configuration. The parameter β is related to the jump in the inclination of the angles $[\varepsilon]$ by the equation

$$\beta=-2\Delta^{-1}[\varepsilon].$$

Sufficient conditions for instability

If $\beta\neq 0$, the complicated form of Eqs (3.1) and (3.2) prevents us from analysing the sign-definiteness of the eigenvalues of the spectral problem in the same volume as was done for $\beta=0$. Here we will give a qualitative discussion and formulate some hypotheses.

We will consider the neutral stability relation (3.2) for fixed χ, $\bar{\alpha}$, s and h as an equation in β. This equation can have: (A) four real roots $\pm\beta_1$, $\pm\beta_2$ (we take $0<\beta_1<\beta_2$), (B) two imaginary roots $\pm\beta_1$ and two real roots $\pm\beta_2$ ($\beta_2>0$). Case (C), in which (3.2) does not have real roots, is also possible.

It follows from (3.1) that $q^{-4}\| D\|\to+\infty$ as $q\to-\infty$. Hence Eq. (3.1) must have a negative root q if β satisfies the inequality $(q^{-4}\| D\|)_{q=0}<0$. In case (A) this inequality is satisfied when $\beta_1<|\beta|<\beta_2$ and in case (B) when $|\beta|<\beta_2$. This means that for the given values of β the periodic structure is unstable.

For $\beta_2<|\beta|$ in cases (A) and (B), and also for any β in case (C), the periodic structure appears to be stable. That assumption is based on the following property of relation (3.1): for any fixed s, $h\neq 0$, χ and $\bar{\alpha}$ there exists a value of β_0, ($\beta_0>0$), such that for any β satisfying $\beta_0<|\beta|$, Eq. (3.1) has no negative roots q.

We will now investigate the stability when $|\beta|<\beta_1$ in case (A). Case (A) can only occur when $d_t d_c>0$, which is satisfied if: (1) $\bar{\alpha}\in(-\infty,\ \min(\bar{\alpha}_1^f,\ \bar{\alpha}_1^g))\cup(\max(\bar{\alpha}_2^f,\ \bar{\alpha}_2^g),\ +\infty)$ or (2) $\bar{\alpha}\in(\max(\bar{\alpha}_1^f,\ \bar{\alpha}_1^g),\ \min(\bar{\alpha}_2^f,\ \bar{\alpha}_2^g))$. It appears that in the first case the state is unstable, and in the second case stable. This hypothesis is based on the result which was obtained when considering states satisfying $\beta=0$.

Asymptotic behaviour of long-period perturbations. Effect of "shear instability"

We put $y=\beta^2$ in (3.2). We will investigate the behaviour of the roots y_1 and y_2 of Eq. (3.2) in the case of long-period perturbations. When $h \to 0$ and $s \in (0, 1)$ the limiting values of the coefficients of this equation have the form

$$a = 0, \quad b = \overline{\alpha}^2 \frac{(1-2s)^2}{s(1-s)} - 4\overline{\alpha}(s\sqrt{\chi} + \frac{1-s}{\sqrt{\chi}}) - 4$$

$$c = -4\overline{\alpha}\overline{\alpha}_{01}^f(\overline{\alpha} - \overline{\alpha}_{01}^f)(\overline{\alpha} - \overline{\alpha}_{01}^g) \tag{6.1}$$

The root y_1 is calculated from the formulae

$$y_1 = \frac{4\overline{\alpha}\overline{\alpha}_{01}^f(\overline{\alpha} - \overline{\alpha}_{01}^f)(\overline{\alpha} - \overline{\alpha}_{01}^g)s(1-s)}{(1-2s)^2(\overline{\alpha} - \overline{\alpha}_+)(\overline{\alpha} - \overline{\alpha}_-)}, \quad s \neq 1/2 \tag{6.2}$$

$$\overline{\alpha}_\pm \equiv \frac{2s(1-s)}{(1-2s)^2}\left(s\sqrt{\chi} + \frac{1-s}{\sqrt{\chi}} \pm \left(\left(s\sqrt{\chi} + \frac{1-s}{\sqrt{\chi}}\right)^2 + \frac{(1-2s)^2}{s(1-s)}\right)^{1/2}\right) \tag{6.3}$$

$$y_1 = \overline{\alpha}(\overline{\alpha} + \tfrac{1}{2}(\sqrt{\chi} + 1/\sqrt{\chi})), \quad s = 1/2 \tag{6.4}$$

When $h \to 0$ we find that $y_2 \to +\infty$ if $\overline{\alpha} < \overline{\alpha}_-$ or $\overline{\alpha}_+ < \overline{\alpha}$, and $y_2 \to -\infty$ if $\overline{\alpha}_- < \overline{\alpha} < \overline{\alpha}_+$.

Different values of y_1 and y_2 give the limiting situations for cases (A), (B) and (C). We have case (A) if $y_1 > 0$ and $y_2 \to \infty$, case (B) if $y_1 > 0$ and $y_2 \to -\infty$ or $y_1 < 0$ and $y_2 \to +\infty$, and case (C) if $y_1 > 0$ and $y_2 \to -\infty$.

We will elucidate the relative positions of the roots $\overline{\alpha}_{01}^f$, $\overline{\alpha}_{01}^g$ and $\overline{\alpha}_\pm$. From formulae (4.1) and (6.3) it is clear that $\overline{\alpha}_{01}^f$, $\overline{\alpha}_{01}^g$ and $\overline{\alpha}_-$ are negative, while $\overline{\alpha}_\pm$ is positive. A more detailed analysis shows the following

1. if $s \geqslant S$ $(S \equiv \sqrt{\chi}/(1+\sqrt{\chi}))$, $\chi \geqslant 1$ or if $s \leqslant S$, $\chi \leqslant 1$, then $\overline{\alpha}_{01}^g \leqslant \overline{\alpha}_{01}^f \leqslant \overline{\alpha}_-$;
2. if $s \geqslant S$, $\chi \leqslant 1$ or if $s \geqslant S$, $\chi \geqslant 1$, then $\overline{\alpha}_{01}^f \leqslant \overline{\alpha}_{01}^g$, $\overline{\alpha}_{01}^f \leqslant \overline{\alpha}_-$;
3. if $s = \tfrac{1}{2}$, then $\overline{\alpha}_+ \to +\infty$, $\overline{\alpha}_- = -2/(\sqrt{\chi} + 1/\sqrt{\chi})$, $\overline{\alpha}_{01}^g = \overline{\alpha}_-$, $\overline{\alpha}_{01}^f = 1/\overline{\alpha}_-$, i.e. $\overline{\alpha}_{01}^f \leqslant -1 \leqslant \overline{\alpha}_- = \overline{\alpha}_{01}^g$.

It is impossible for the inequalities $\overline{\alpha}_- < \overline{\alpha}_{01}^g$ and $\overline{\alpha}_- < \overline{\alpha}_{01}^f$ to be satisfied simultaneously.

Knowing the possible positions of the roots $\overline{\alpha}_{01}^f$, $\overline{\alpha}_{01}^g$ and $\overline{\alpha}_\pm$ along the $\overline{\alpha}$ axis, we qualitatively construct graphs of the behaviour of the critical value of the parameter β as a function of $\overline{\alpha}$: $\beta = \sqrt{(y_1)}$ (Fig. 2). In Fig. 2 regions of instability are denoted by I and regions where the sufficient condition for instability is not satisfied are denoted by S. There is a justification for assuming that in the S regions the structure is stable (see the analysis of cases (A)–(C) given above). From the graphs it is clear that increasing the absolute value of $\overline{\alpha}$ leads to instability. Increasing the absolute value of β can stabilize the system in a range of cases. This picture is qualitatively identical with the results of investigations of the local stability of coherent boundaries [1–3]. However, here a qualitatively new effect occurs: if $\overline{\alpha} \in (\max(\overline{\alpha}_{01}^f, \overline{\alpha}_{01}^g), \overline{\alpha}_-)$, then as $|\beta|$ increases the system goes from the stable domain to the unstable domain. The value of $|\beta|$ can be increased through the shear stresses σ_{12}^+. Hence it is natural to call this effect the "shear instability effect of a periodic two-phase structure". "Shear instability" also occurs in the problem of the plane nucleation of a new phase in an infinite elastic matrix [5].

In the context of asymptotically long-period perturbations we will consider the limiting cases $\chi \to \infty$, $\chi \to 0$ and $\chi = 1$.

From formulae (4.1) and (6.3) it is clear that $\overline{\alpha}_+ \to +\infty$, $\overline{\alpha}_- \to 0$, $\overline{\alpha}_{01}^g \to 0$, $\overline{\alpha}_{01}^f \to -\infty$ if $\chi \to \infty$ or $\chi \to 0$. For $\chi \to 0$, we have $\overline{\alpha}_-/\overline{\alpha}_{01}^g \to s/(1-s)$. This means that $\overline{\alpha}_{01}^g > \overline{\alpha}_-$ when $s > \tfrac{1}{2}$ and $\overline{\alpha}_{01}^g < \overline{\alpha}_-$ when $s < \tfrac{1}{2}$. For $\chi \to \infty$ we have $\overline{\alpha}_-/\overline{\alpha}_{01}^g \to (1-s)s$. This means that $\overline{\alpha}_{01}^g < \overline{\alpha}_-$ when $s > \tfrac{1}{2}$ and $\overline{\alpha}_{01}^g > \overline{\alpha}_-$ when $s < \tfrac{1}{2}$. Consequently, in a periodic structure (with $\chi \gg 1$ or $\chi \ll 1$) the "shear instability" effect must occur when the concentration of the less stiff phase exceeds the concentration of the stiffer phase.

For $\chi = 1$ we have $\overline{\alpha}_{01}^f = \overline{\alpha}_{01}^g = -1$, $-1 \leqslant \overline{\alpha}_- \leqslant 0$. We note that for $\chi = 1$ the parameters $\overline{\alpha}$ and β depend only on the form of the ND transformation tensor $\Delta_{(ij)}$ and do not depend on σ_{ij}. This means that one cannot influence the stability by changing the stressed state. In this case for

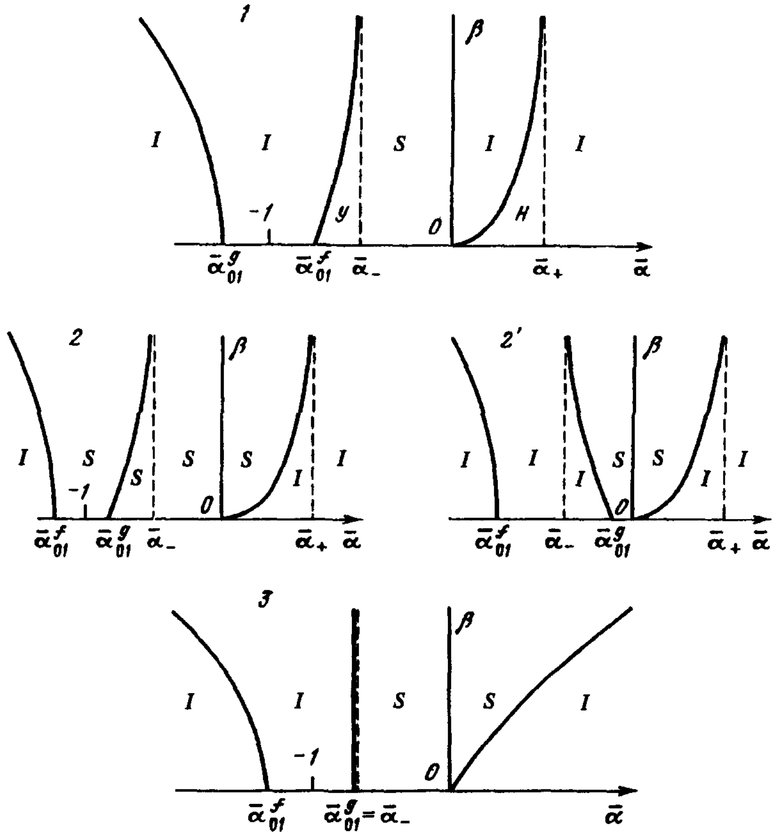

FIG. 2.

$-1 < \bar{\alpha} < \bar{\alpha}_-$ the system will be stable if $|\Delta_{(12)}|$ is smaller than some critical value, and unstable if $|\Delta_{(12)}|$ is greater than that critical value. Thus one can talk of "shear instability" due to the "natural" shear deformation.

We shall analyse the dependence of the critical values of the parameter β on the concentration s. It follows from (6.2) that these values $\beta(s)$ depend strongly on the position of the parameter $\bar{\alpha}$ relative to $\bar{\alpha}^f_{01}$, $\bar{\alpha}^g_{01}$ and $\bar{\alpha}_\pm$. Figure 3 shows graphs of $\bar{\alpha}^f_{01}(s)$ (the line LL') $\bar{\alpha}^g_{01}(s)$ (the curve MM'), $\bar{\alpha}_+(s)$ (the curves OK and $K'N$), $\bar{\alpha}_-(s)$ (the curve $ON'N$) and $\bar{\alpha}^g_{02}(s)$ (the line ON) for $\chi = 0$ and 1. Regions corresponding to the three cases (A)–(C) are marked (bearing in mind limiting situations corresponding to asymptotically long-period perturbations), and those regions where "shear instability" occur are marked with an asterisk. Analysis of these graphs enables us to represent all possible forms of $\beta(s)$ for various values of $\bar{\alpha}$.

We fix the value of the parameter $\bar{\alpha}$ $(\bar{\alpha} = \bar{\alpha}')$. The line $\bar{\alpha} = \bar{\alpha}'$ intersects the curves $\bar{\alpha}_\pm(s)$ at the points s_1 and s_2, and the curves $\bar{\alpha}^f_{01}(s)$ and $\bar{\alpha}^g_{01}(s)$ at the points s_1^0 and s_2^0. (As can be seen from the graphs, these points do not always exist.) The concentration values s_1 and s_2 correspond to the vertical asymptotes of the function $\beta(s)$, while s_1^0 and s_2^0 are zeros of $\beta(s)$. Graphs of $\beta(s)$ for various fixed $\bar{\alpha}$ when $\chi = 0, 1$ are shown in Fig. 4 $(\beta(s) = \sqrt{y_1})$. Using the analysis performed above for cases (A)–(C) one can find in Fig. 4 the stable and unstable domains.

Consider an example. We will denote the points of intersection of the curves $\bar{\alpha}_-(s)$ and $\bar{\alpha}^g_{01}(s)$ in Fig. 3 by $\bar{\alpha}'$ and $\bar{\alpha}''$ $(\bar{\alpha}'' < \bar{\alpha}')$. Suppose $\bar{\alpha}' \in (\bar{\alpha}'', \bar{\alpha}')$. Then for $s \in [0, s_1^0] \cup [s_2, 1]$ the two-phase structure is unstable for all β, for $s \in (s_1^0, s_1)$ we have "shear instability", for $s \in [s_1, s_2^0]$ the system is stable for all β, and for $s \in (s_2^0, s_2)$ the system can become stable when $|\beta|$ is increased (Fig. 4b, $\bar{\alpha}' = -0.4$).

The above analysis shows, in particular, that when $s \to 0$ or $s \to 1$ the periodic structure cannot be stable with respect to long-period perturbations for any values of the stressed-state parameters $\bar{\alpha}$ and β.

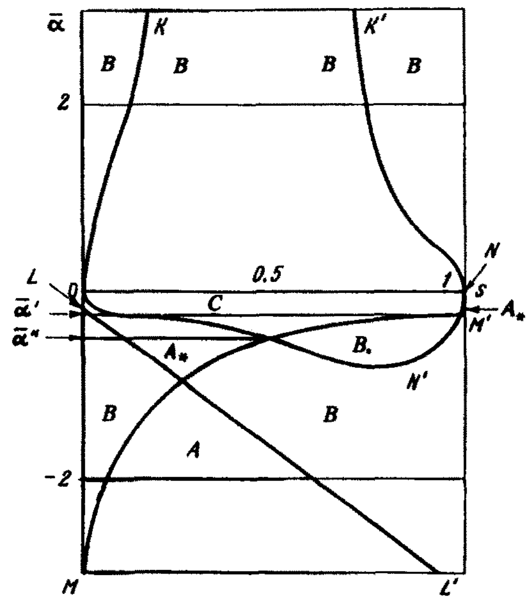

FIG. 3.

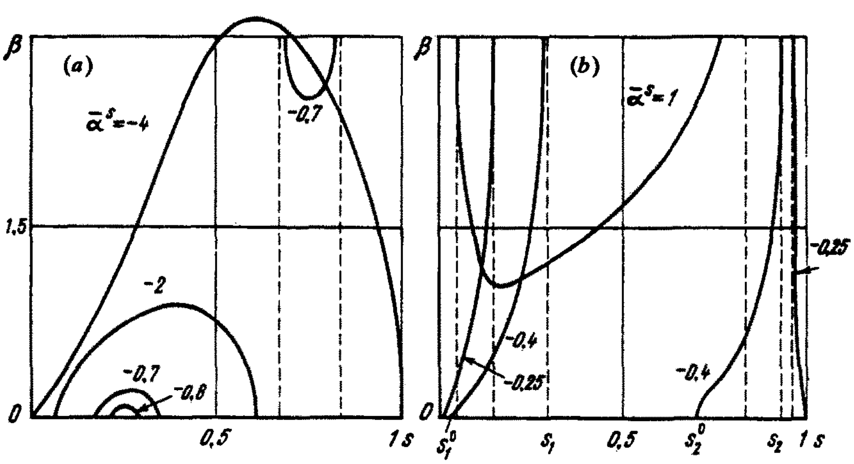

FIG. 4.

The asymptotic form of small concentrations

If $s \to 0$ (or $s \to 1$) and $h \to 0$, then in Eq. (3.2) we have $a/b \to 0$, $d/b \to 0$ and, consequently, $\bar{\alpha}^2\beta^2 \to 0$. Hence the neutral stability curves in the $(\bar{\alpha}, \beta)$ turn into the $\bar{\alpha}$ and β axes. In other words, when $s \to 0$ (or $s \to 1$) the stable state region vanishes asymptotically. Consequently, a two-phase structure with thin periodic nucleation surfaces is unstable for all $\bar{\alpha}$, β and h.

We note two more asymptotic cases. When $h_+ \sim h_- \to +\infty$ we arrive at the solution of the stability problem for a coherent boundary between half-spaces [2]. As $h \to +\infty$, $h_- \sim 1$, we obtain the solution to the stability problem for a plane nucleation layer for the new phase in an infinite elastic matrix of the original phase [5].

Results of numerical calculations

The roots β_1 and β_2 of the neutral stability equation (3.2) were calculated on a computer for various values of the parameters $\bar{\alpha}$, s, h and χ. For $\chi=0.1$, $h=2$, Figs 5 and 6 show graphs of the dependence of the critical values of the parameter β (i.e. the roots β_1 and β_2) on $\bar{\alpha}$ and s for the half-plane $\beta\geqslant 0$. The continuation to the half-plane $\beta\leqslant 0$ is found using symmetry. The stable and unstable regions can be marked out using the above analysis of cases (A)–(C).

In the local stability analysis of a single coherent boundary in [2], two branches of the neutral stability hyperbola were found ([2], Fig. 1). As can be seen in Fig. 6, each of the branches for $h<+\infty$ is replaced by two new curves. When $h\to+\infty$, these curves become a hyperbola [2], while for $h\to 0$ they become the curves shown in Fig. 2. Here the qualitative difference between the stability conditions for a single plane new phase nucleation layer in an infinite elastic matrix [5] and a two-phase periodic structure of nucleation layers unstable with respect to long-period perturbations becomes manifest, and the periodic structure, as was shown above, can be stable with respect to such perturbations.

If in case (A) for some $\bar{\alpha}$, h and s the discriminant of Eq. (3.2) vanishes, then $\beta_1=\beta_2$: in the parameter space of $\bar{\alpha}$, h, s and β the hypersurface $\beta=\beta_1(\bar{\alpha}, h, s)$ turns into the hypersurface $\beta=\beta_2(\bar{\alpha}, h, s)$. The values of $\bar{\alpha}$, h and s indicated in Figs 5 and 6 governs the points of inflection of the curves $\beta(s)$ and $\beta(\bar{\alpha})$. The points of inflection are indicated by s_* and $\bar{\alpha}_*$.

For $s\in[s^0, s_*]$, $\bar{\alpha}=-0.8$ and for $\bar{\alpha}\in[\alpha_{01}^{\xi}, \bar{\alpha}_*]$, $s=0.1$, 0.5, 0.9 the branches of the critical value curves of the parameter β, respectively, determine the two-valued functions $\beta(z)$ and $\beta(\bar{\alpha})$, and when $0\leqslant\beta\leqslant\beta_1$ the system is stable (see the analysis of case A), while for $\beta_1\leqslant\beta\leqslant\beta_2$ it is unstable, i.e. increasing $|\beta|$ in these cases causes instability. This means that here the "shear instability" of the two-phase periodic structure occurs with respect to perturbations with finite period $(0<h<+\infty)$. In the case under consideration, as opposed to the long-period perturbation case, the system becomes stable for $\beta_2<\bar{\beta}$ — here the stabilizing role of the parameter β becomes manifest (for $h\to 0$ we had $\beta_2^2\to\infty$) [2, 3].

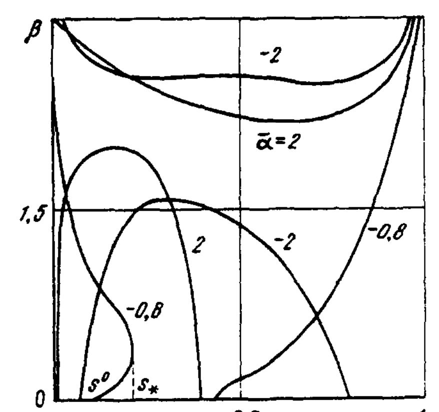

FIG. 5.

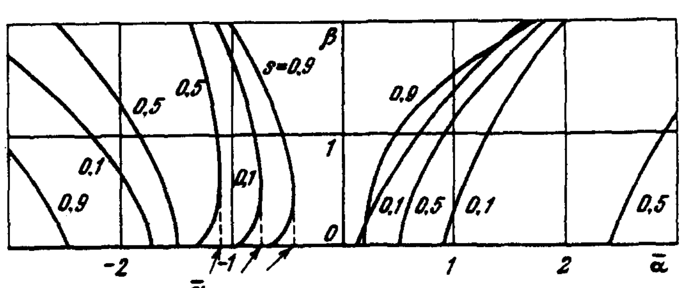

FIG. 6.

The effect of "shear instability" is a distinguishing feature of layered systems with coherent interphase boundaries. A similar effect is impossible in layered systems found in states of plane homogeneous deformation and containing phase-transformation surfaces with slippage [1], because for non-zero shear stress components σ_{12} the equilibrium condition is not satisfied.

I wish to thank M. A. Grinfel'd for numerous discussions.

REFERENCES

1. GRINFEL'D M. A., *Methods of Continuum Mechanics in the Theory of Phase Transformations.* Nauka, Moscow, 1990.
2. GRINFEL'D M. A. and LAZAREV M. P., The sufficient conditions for the instability of coherent interphase boundaries. *Dokl. Akad. Nauk SSSR* **300**, 4, 836–840, 1988.
3. GRINFEL'D M. A. and LAZAREV M. P., Stability of heterogeneous systems with coherent interphase boundaries. *Prikl. Mat. Mekh.* **53**, 2, 284–293, 1989.
4. LOVE A., A *Treatise on the Mathematical Theory of Elasticity.* Dover, New York, 1927.
5. LAZAREV M. P., The problem of the stability of a layer of a new phase in an unbounded elastic matrix. *Dokl. Akad. Nauk SSSR* **317**, 5, 1066–1070, 1991.

Translated by R.L.Z.